Fermi liquid theory

Subir Sachdev

Department of Physics, Harvard University, Cambridge, Massachusetts, 02138, USA and
Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada

(Dated: April 4, 2016)

Abstract

We describes the central ideas in the quasiparticle theory of metals. All low energy properties can be described by the dynamics of long-lived quasiparticles near a Fermi surface in momentum space. A non-perturbative, “topological” proof is given of the Luttinger relation between the volume enclosed by the Fermi surface and the density of all particles.
I. FERMI LIQUIDS

Fermi Liquid Theory

Free electron gas

\[ H = \sum_p \left( \frac{\hbar^2 k^2}{2m} - \mu \right) c_p^+ c_p \]

(ignoring spin)

\[ = \sum_p \varepsilon_p c_p^+ c_p. \]

Ground state

\[ |G\rangle = \prod_{\varepsilon_p < 0} c_p^+ |0\rangle. \]

\[ \varepsilon_p = 0 \rightarrow \text{Fermi surface (FS)} \]

Excited states

Pandas (inside FS) \[ c_p^+ |G\rangle \text{ with } p \text{ inside FS} \]

Excitation energy \[ \varepsilon_p > 0. \]

Holes \[ c_p |G\rangle \text{ with } p \text{ outside FS} \]

Excitation energy \[ -\varepsilon_p > 0. \]
Assume some is true for interacting system i.e. the low energy excitations are quasiparticles and quasiholes whose energies vanish on some Fermi surface. These are well-defined only near the F5. However, it is tedious to keep track of quasiparticles and quasiholes separately. So for convenience we postulate that the ground state is made up of quasiparticles with their number $N(p)$ given by
Then the excited states are described by the changes

\[ S_n(p) = n(p) - n_0(p) \]

in the distribution functions.

The energy of an excited state is written as

\[ E[S_{np}] = \sum E_p S_{np} + \frac{1}{2} \sum F_{pp'} S_{np} S_{np'} \]

Landau parameters.

Both terms are of order \( (P-P^0)^2 \)

where \( |P-P^0| \) is the distance which
quasiparticles are excited.

We usually write

$$
\epsilon_p = v_F (p - p_F)
$$

with

$$
v_F = \frac{p_F}{m^*}
$$

Specific Heat

Equal numbers of particles and holes are excited near the FS at low T, and so

$$
\sum_p \Delta E_p \rightarrow 0.
$$

So Fermi parameters make no difference,

and the specific heat is that of the free electron gas with $m \rightarrow m^*$

$$
C_V = \frac{\pi^2 k_B^2}{3} N(0)
$$

$$
N(0) = \frac{m^* p_F}{2 \pi^2 k_F^3} \text{ in } d = 3.
$$
Compressibility

Without Landau parameters

\[ H = \sum_p \epsilon_p c_p^+ c_p \]

\[ \rightarrow \sum_p \epsilon_p c_p^+ c_p - \delta \mu \sum_p c_p^+ c_p. \]

So \( S_n(p) = \frac{2}{\delta \mu} \sum_p \langle c_p^+ c_p \rangle \delta \mu \)

\[ = \frac{2}{\delta \mu} \sum_p f(\epsilon_p - \delta \mu) \delta \mu \]

\[ = \delta \mu \int (-2f) = \delta \mu \delta(\epsilon_p). \]

Then \( \sum S_n(p) = \delta \mu N(0) \)

So \( \frac{dn}{d\mu} = N(0) \)
With Landauer parameters

\[ E_p \rightarrow E_p + \sum_{p'} F_{pp'} S_{np'} \]

So

\[ S_{np} = S \left[ f (E_p - \mu + \sum_{p'} F_{pp'} S_{np}) \right] \]

and

\[ \Delta \approx N(0) \]

\[ = S_{np} S(E_p) - S(E_p) \sum_{p'} F_{pp'} S_{np} \]

So let us write

\[ S_{np}(p) = AS(E_p) \]

Then

\[ A = S_{np} - F_0 A \]

where

\[ F_0 = \sum_{p'} S(E_{p'}) \]

\[ \Rightarrow A = \frac{S_{np}}{1 + F_0} \]

\[ \Rightarrow \frac{dn}{d\mu} = \frac{N(0)}{1 + F_0} . \]
II. QUASIPARTICLE STABILITY

\[ G(k, i\omega_n) = \frac{Z}{i\omega_n - V_F (k - k_F)} \]

where \( V_F = \frac{k_F}{m^*} \)

and \( 2 \int \frac{d^3k}{8\pi^3} = N \) (Luttinger relation).

**Beyond Hartree-Fock**

\( Z \) and \( m^* \) will have corrections at order \( U^2 \).

However, more significant is the imaginary of the self-energy, representing decay of particle excitations.
By Fermi's golden rule decay rate for a particle with momentum $k$.

$\frac{1}{\tau_k} = |U|^2 \int \frac{d^{3}k'}{V} \int_{k'} \frac{d^{3}k''}{V} \frac{1}{\epsilon_k} \left( 1 - f(\epsilon_{k'+q}) \right) f(\epsilon_{k'}) \left( 1 - f(\epsilon_{k''-q}) \right)

\sim \frac{|U|^2}{\tau_k} \left[ \rho(E_F) \right]^3 \int_{-\infty}^{\infty} d\epsilon' \int_{-\infty}^{\infty} d\epsilon'' \Theta(\epsilon_k + \epsilon' - \epsilon'') \epsilon_k^2 \epsilon' \epsilon''

So lifetime $\tau \rightarrow \infty \sim \frac{1}{\epsilon^2} \sim \epsilon \rightarrow 0$. Particle decay effects can be neglected near the Fermi surface!
Scattering rate of fermions vanishes on the Fermi surface.

\[ \Rightarrow \text{Im } \Sigma(k^F, \omega \to 0) = 0. \]

\[ \Rightarrow G^{-1}(k = k^F, \omega = 0) = 0 \]

defines the Fermi surface for an interacting system.

In a Fermi liquid, we have

\[ G(k, i\omega) = \frac{Z}{i\omega - V_F(k + k^F) + i\omega^2 \text{sgn}(\omega)} \]

As \( k \to k^F \) and \( \omega \to 0 \).

Use this to compute

\[ n(k) = \frac{1}{2\pi} \int \frac{d\omega}{Z} G(k, i\omega) e^{i\omega_0^+} \]

Then

\[ n(k) = \frac{1}{Z} \]

\[ k \]
The relationship \( n_F \)

\[ N = \oint 2 \int \frac{d^3 k}{8 \pi^3} \] is exact!

Write

\[ N = +2 \int \frac{d^3 k}{8 \pi^3} \frac{dw}{2 \pi} G(k, iw) e^{iw \omega} \]

\[ = +2i \int \frac{d^3 k}{8 \pi^3} \frac{dw}{2 \pi} \left[ i G_k(iw) \frac{2}{2 \omega} \sum_k (iw) \right. \]

\[ - \frac{2}{2 \omega} \ln G(iw) \] \]

Where \( G(iw) = \frac{1}{iw - \epsilon_k - \sum_k (iw)} \)

We will show below that

\[ \int \frac{d^3 k}{8 \pi^3} \frac{dw}{2 \pi} G(iw) \frac{2}{2 \omega} \sum_k (iw) = 0 \] \( (*) \).
So
\[ N = -2i \int \frac{d^3k \ e^{i \omega t}}{(2\pi)^3} \int_0^\infty \frac{dZ}{2\pi} \ln \frac{G_k (Z+i0^+)}{G_k (Z+i0^-)}. \]

\[ = -2i \int \frac{d^3k}{(2\pi)^3} \ln \frac{G_k (i0^+)}{G_k (i0^-)}. \]

\[ = 2 \int \frac{d^3k}{(2\pi)^3} \Theta (-\varepsilon_k - \Sigma_k (i0^+)) \]

\[ = 2 \int \frac{d^3k}{8\pi^3} \]

where \( k_F \) is defined by
\[ G^{-1} (k = k_F, \omega = 0) = 0. \]

The location of Fermi surface.
Finally we need to establish $\Phi (A)$.

$(A)$ can be shown if there exists a functional $\Phi [G_k (i \omega)]$ (Luttinger-Ward functional) with 2 properties

(i) $\sum_k (i \omega) = \frac{\delta \Phi}{G_k (i \omega)}$

(ii) $\Phi [G_k (i \omega + i \varepsilon)] = \Phi [G_k (i \omega)]$

\[ \Phi = \sum \text{diagram} + \ldots \]

Sum of 2 perturbative irreducible graphs.

For more details see cond-mat/0406671.
We take $N$ particles, each with charge $Q$, on a $L_x \times L_y$ lattice on a torus. We pierce flux $\Phi = \hbar c / Q$ through a hole of the torus. An exact computation shows that the change in crystal momentum of the many-body state due to flux piercing is

$$P_{xf} - P_{xi} = \frac{2\pi N}{L_x} (\text{mod } 2\pi) = 2\pi \nu L_y (\text{mod } 2\pi)$$

where $\nu = N / (L_x L_y)$ is the density.
Topology and the Fermi surface size

Proof of
\[ P_{xf} - P_{xi} = \frac{2\pi N}{L_x} (\text{mod } 2\pi) = 2\pi \nu L_y (\text{mod } 2\pi). \]

The initial and final Hamiltonians are related by a gauge transformation

\[ U_G H_f U_G^{-1} = H_i , \quad U_G = \exp \left( i \frac{2\pi}{L_x} \sum_i x_i \hat{n}_i \right). \]

while the wavefunction evolves from \( |\Psi_i\rangle \) to \( U_T |\Psi_i\rangle \), where \( U_T \) is the time evolution operator. We want to work in a fixed gauge in which the initial and final Hamiltonians are the same: in this gauge, the final state is \( |\Psi_f\rangle = U_G U_T |\Psi_i\rangle \). Let \( \hat{T}_x \) be the lattice translation operator. Then we can establish the above result using the definitions

\[ \hat{T}_x |\Psi_i\rangle = e^{-iP_{zi}} |\Psi_i\rangle , \quad \hat{T}_x |\Psi_f\rangle = e^{-iP_{zf}} |\Psi_f\rangle , \]

and the easily established properties

\[ \hat{T}_x U_T = U_T \hat{T}_x , \quad \hat{T}_x U_G = \exp \left( -i \frac{2\pi N}{L_x} \right) U_G \hat{T}_x \]
Topology and the Fermi surface size

\[ \Delta P_x = 2\pi \nu L_y \pmod{2\pi}, \quad \Delta P_y = 2\pi \nu L_x \pmod{2\pi} \]

Now we compute the momentum balance assuming that the only low energy excitations are quasiparticles near the Fermi surface, and these react like free particles to a sufficiently slow flux insertion. So each quasiparticle picks up a momentum \( \delta \vec{p} \equiv (2\pi/L_x, 0) \), and then we can write (with \( \delta n_p \) the quasiparticle density excited by the flux insertion)

\[ \Delta P_x = \sum_p \delta n_p p_x. \]
Now we compute the momentum balance assuming that the only low energy excitations are quasiparticles near the Fermi surface, and these react like free particles to a sufficiently slow flux insertion. So each quasiparticle picks up a momentum $p(2\frac{x}{L},0)$, and then we can write (with $n_p$ the quasiparticle density excited by the flux insertion)

$$P_x = p n_p p_x.$$  

Now $n_p = \pm 1$ on a shell of thickness $\delta p \cdot d\vec{S}_p$ on the Fermi surface (where $\vec{S}_p$ is an area element on the Fermi surface). So we can write the above as a surface integral

$$\Delta P_x = \int_{FS} p_x \left( \frac{L_x L_y}{4\pi^2} \right) \delta p \cdot d\vec{S}_p$$

by the divergence theorem. So

$$\Delta P_x = (\delta p \cdot \hat{x}) \int_{FV} \left( \frac{L_x L_y}{4\pi^2} \right) dV$$

Now $\delta n_p = \pm 1$ on a shell of thickness $\delta p \cdot d\vec{S}_p$ on the Fermi surface (where $\vec{S}_p$ is an area element on the Fermi surface). So we can write the above as a surface integral

$$\Delta P_x = 2\pi \nu L_y (mod \ 2\pi), \quad \Delta P_y = 2\pi \nu L_x (mod \ 2\pi)$$

### Topology and the Fermi surface size

Now $\delta n_p = \pm 1$ on a shell of thickness $\delta p \cdot d\vec{S}_p$ on the Fermi surface (where $\vec{S}_p$ is an area element on the Fermi surface). So we can write the above as a surface integral

$$\Delta P_x = \int_{FS} p_x \left( \frac{L_x L_y}{4\pi^2} \right) \delta p \cdot d\vec{S}_p$$

by the divergence theorem. So
Topology and the Fermi surface size

\[ \Delta P_x = 2\pi \nu L_y (\text{mod } 2\pi) , \quad \Delta P_y = 2\pi \nu L_x (\text{mod } 2\pi) \]

\[ \Delta P_x = \left( \frac{2\pi}{L_x} \right) \frac{L_x L_y}{4\pi^2} V_{FS} , \quad \Delta P_y = \left( \frac{2\pi}{L_y} \right) \frac{L_x L_y}{4\pi^2} V_{FS} \]

where \( V_{FS} \) is the volume of the Fermi surface. So, although the quasiparticles are only defined near the Fermi surface, by using Gauss’s Law on the momentum acquired by quasiparticles near the Fermi surface, we have converted the answer to an integral over the volume enclosed by the Fermi surface.

Now we equate these values to those obtained above, and obtain

\[ N - L_x L_y \frac{V_{FS}}{4\pi^2} = L_x m_x , \quad N - L_x L_y \frac{V_{FS}}{4\pi^2} = L_y m_y \]

for some integers \( m_x, m_y \). Now choose \( L_x, L_y \) mutually prime integers; then \( m_x L_x = m_y L_y \) implies that \( m_x L_x = m_y L_y = pL_x L_y \) for some integer \( p \). Then we obtain

\[ \nu = \frac{N}{L_x L_y} = \frac{V_{FS}}{4\pi^2} + p. \]

This is Luttinger’s theorem.