Quantum Phase Transitions

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Chapter 4: *Exercises*

1. **Fluctuation dissipation theorem.** We showed in class that the spin correlation function of the classical $D = 2$ Ising model was simply related to the response function

$$\chi_{ij}(\omega_n) = \int_0^{1/T} d\tau e^{i\omega_n \tau} \langle \hat{\sigma}_i^z(\tau) \hat{\sigma}_j^z(0) \rangle \tag{1}$$

where $\omega_n = 2\pi n T$. Evaluate this correlation function in terms of the exact eigenstates of H_I, $H_I|m\rangle = E_m|m\rangle$. By inserting the completeness identity, $1 = \sum_m |m\rangle\langle m|$ around the $\hat{\sigma}^z$ operators, show that

$$\chi_{ij}(\omega_n) = \frac{1}{Z} \sum_{m,m'} \langle m'|\hat{\sigma}_i^z|m\rangle\langle m|\hat{\sigma}_j^z|m'\rangle \frac{e^{-E_m/T} - e^{-E_{m'}/T}}{i\omega_n - E_m + E_{m'}} \tag{2}$$

where $Z = \sum_m e^{-E_m/T}$ is the partition function. Hence show that

$$\chi_{ij}(\omega_n) = \int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \frac{\rho_{ij}(\Omega)}{\Omega - i\omega_n} \tag{3}$$

where

$$\rho_{ij}(\Omega) = \frac{\pi}{Z} \sum_{m,m'} \langle m'|\hat{\sigma}_i^z|m\rangle\langle m|\hat{\sigma}_j^z|m'\rangle (e^{-E_{m'}/T} - e^{-E_m/T}) \delta(\Omega - E_m + E_{m'}) \tag{4}$$

Similarly, express the dynamic structure factor

$$S_{ij}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \hat{\sigma}_i^z(t) \hat{\sigma}_j^z(0) \rangle \tag{5}$$

in terms of exact eigenstates and show that

$$S_{ij}(\omega) = \frac{2}{1 - e^{-\omega/T}} \rho_{ij}(\omega) \tag{6}$$

2. **Linear response theory.** Consider the response of the system described by H_I to a time-dependent external magnetic field $h_i(t)$ under which

$$H_I \to H_I - \sum_i h_i(t) \hat{\sigma}_i^z \tag{7}$$

As shown in practically any text book on many body theory (*e.g.* Fetter and Walecka), we can obtain the linear response to this external perturbation simply by integrating the Schroedinger equation order by order in h_i. To first order in h_i, the result is

$$\delta\langle \hat{\sigma}_i^z \rangle(t) = \sum_j \int_{-\infty}^{\infty} dt' \chi_{ij}(t - t') h_j(t') \tag{8}$$

where the initial δ indicates 'change due to external field' and

$$\chi_{ij}(t-t') = i\theta(t-t')\langle[\hat{\sigma}_i^z(t), \hat{\sigma}_j^z(t')]\rangle. \tag{9}$$

Now θ is a step function which imposes causality, and $[\cdot,\cdot]$ represents the commutator of operators in the Heisenberg representation. Again after inserting complete sets of exact eigenstates, show that the Fourier transform of χ_{ij}

$$\chi_{ij}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t}\chi_{ij}(t) \tag{10}$$

can be written as

$$\chi_{ij}(\omega) = \int_{-\infty}^{\infty} \frac{d\Omega}{\pi}\frac{\rho_{ij}(\Omega)}{\Omega - \omega - i\eta} \tag{11}$$

where η is a positive infinitesimal. So $\chi_{ij}(\omega)$ is obtained from $\chi_{ij}(\omega_n)$ by analytically continuing the latter from the imaginary frequency axes to points just above the real frequency axis.

3. This problem considers various properties of the Ising chain in a transverse field in (4.1)

   (a) First, consider the limit $g \ll 1$. Write down the ground state wavefunction, with the spins mostly up, correct to first order in g.

   (b) Use this wavefunction to compute $N_0 = \langle\hat{\sigma}^z\rangle$ to second order in g. Don't forget to properly normalize the wavefunction. It is useful to carry out the computation for M sites with periodic boundary conditions; intermediate steps will include factors of M, but all M dependence should cancel out in the final answer.

   (c) With the dispersion relation (4.22), compute the range of energies, as a function of total momentum, over which the two-particle continuum exists

   (d) Now consider the opposite limit of $g \gg 1$. Here we will use the exact solution obtained by the Jordan Wigner transformation. The ground state $|G\rangle$ satisfies $\gamma_k|G\rangle = 0$ and the state with a single quasiparticle is $|k\rangle = \gamma_k^\dagger|G\rangle$. The weight of the delta function peak in the dynamic structure factor is determined by the quasi-particle residue $Z = e^{-ikr_i}\langle G|\hat{\sigma}_i^z|k\rangle$. Because of the non-locality of the relationship (4.29) this matrix element is very difficult to evaluate. However, simplifications do occur in the large g limit, where notice from (4.36) that $v_k \to 0$. As a result, the number of c_k fermions in the wavefunctions is small. Use such a method to compute Z to order $1/g^2$.

4. Provide the missing steps leading to the results (4.61) and (4.62).

5. Generalize (4.1) to include also a second-neighbor exchange $-J_2 \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+2}^z$. Determine the dispersion spectrum of the domain wall excitation to lowest order in g. Also consider the limit of large g, and determine the dispersion spectrum of a 'flipped-spin' excitation.

6. We will consider the splitting of the degeneracy in the two-particle subspace defined by the states in (4.15) to first order in $1/g$. Let us write an arbitrary eigenstate, $|\alpha\rangle$ (with energy E_α) in this subspace in the form

$$|\alpha\rangle = \sum_{i>j} \Psi_\alpha(i,j)|i,j\rangle \tag{12}$$

Actually by double-counting, we can rewrite the above as

$$|\alpha\rangle = \sum_{i,j} \Psi_\alpha(i,j)|i,j\rangle \tag{13}$$

where we define $\Psi_\alpha(i,j) = \Psi_\alpha(j,i)$ and $\Psi_\alpha(i,i) = 0$. So we can view Ψ_α as the wavefunction of two bosons hopping on the lattice with a hard core repulsion. For the model H_I (no second neighbor exchange), and to first order in $1/g$, obtain the Schroedinger equation satisfied by $\Psi_\alpha(i,j)$. The translational invariance of the problem implies that we can quite generally write down $\Psi_\alpha(i,j)$ in the form

$$\Psi_\alpha(i,j) = e^{iK(x_i+x_j)}\psi_\alpha(i-j) \tag{14}$$

where K is the center of mass momentum and $\psi_\alpha(i) = \psi_\alpha(-i)$ is the relative wavefunction with $\psi_\alpha(0) = 0$. Obtain the Schroedinger equation obeyed by $\psi_\alpha(i)$. Show that this equation has the very simple solution $\psi_\alpha(i) = \sin(k|i|)$. By inserting this solution back in (13,14) establish (4.17).