The problems below refer to the $\phi^4$ field theory, defined by the partition function ($\alpha = 1 \ldots N$).

$$
Z = \int \mathcal{D}\phi_\alpha(x) \exp \left( -\int d^D x \mathcal{L} \right)
$$

$$
\mathcal{L} = \frac{1}{2} \left[ (\nabla \phi_\alpha)^2 + r\phi_\alpha^2 \right] + \frac{u}{24}(\phi_\alpha^2)^2
$$

1. In the paramagnetic phase, rotational invariance implies that we can write for the susceptibility, $\chi(q)\delta_{\alpha\beta} = \langle \phi_\alpha(q)\phi_\alpha(-q) \rangle$, where $q$ is a $D$-dimensional spacetime momentum. Also, Dyson’s equation has the form $\chi^{-1}(q) = q^2 + r - \Sigma(q)$. Obtain the perturbative expansion for $\Sigma(q)$ to order $u^2$. Leave the result in the form of integrals over momenta.

2. Another useful identity in the theory of Gaussian integrals is

$$
\prod_{i=1}^n \int_{-\infty}^{\infty} \frac{dx_i}{\sqrt{\pi}} \exp \left( -\frac{1}{2} \sum_{i,j} x_i M_{ij} x_j \right) = (\det M)^{-1/2} = \exp \left( -\frac{1}{2} \text{Tr} \ln M \right)
$$

where $M$ is a real, symmetric, positive-definite matrix (i.e. all eigenvalues are positive). This identity can be easily established by changing variables of integration to a basis in which $M$ is diagonal. We will use this identity to compute the free energy density $F$, defined by $Z = \exp(-VF)$ where $V$ is the volume of spacetime. In the paramagnetic phase, $r > 0$, the perturbative expansion for $F$ takes the form $F = C_1 + C_2 u + \mathcal{O}(u^2) + \ldots$, while in the magnetically ordered phase, $r < 0$, it takes the form $F = C_3/u + C_4 + \mathcal{O}(u)$. Obtain expressions for $C_{1-4}$. Assume we have normalized the $\mathcal{D}\phi_\alpha$ in $Z$ to absorb the factor of $1/\sqrt{\pi}$ in (2).

3. This is adapted from Problem (6.5a-c) in Plischke and Bergersen to the notation we are using. You may follow their approach if you wish. We consider the consequences of anisotropy in the $O(N)$ symmetry of $\mathcal{L}$. In some applications to classical ferromagnets and quantum antiferromagnets (which correspond to the case $N = 3$), spin-orbit interactions may introduce a weak anisotropy in which the $r\phi_\alpha^2$ term in $\mathcal{L}$ is replaced.
by
\[ r_s \sum_{\alpha < N} \phi_\alpha^2 + r_n \phi_N^2, \]  
while the quartic term is replaced by
\[ \frac{u_1}{24} \sum_{\alpha, \beta < N} \phi_\alpha^2 \phi_\beta^2 + \frac{u_2}{12} \sum_{\alpha < N} \phi_\alpha^2 \phi_N^2 + \frac{u_3}{24} \phi_N^4. \]  
Clearly, the original problem with full \( O(N) \) symmetry is the case \( r_s = r_n \) and \( u_1 = u_2 = u_3 \). The model with \( r_s = \infty \), \( u_1 = u_2 = 0 \) is the field theory of the Ising model, while the model with \( O(N - 1) \) symmetry is \( r_n = \infty \), \( u_1 = u_3 = 0 \).

(a) Show that the one-loop RG flow equations for this model are:
\[
\begin{align*}
\frac{dr_s}{d\ell} &= 2 r_s + \frac{(N + 1)}{6(1 + r_s)} K u_1 + \frac{1}{6(1 + r_n)} K u_2, \\
\frac{dr_n}{d\ell} &= 2 r_n + \frac{(N - 1)}{6(1 + r_s)} K u_2 + \frac{1}{2(1 + r_n)} K u_3, \\
\frac{du_1}{d\ell} &= \epsilon u_1 - \frac{(N + 1)}{6(1 + r_n)^2} K u_1^2 - \frac{1}{6(1 + r_n)^2} K u_2^2, \\
\frac{du_2}{d\ell} &= \epsilon u_2 - \frac{2}{3(1 + r_s)(1 + r_n)} K u_2^2 - \frac{(N + 1)}{6(1 + r_s)^2} K u_1 u_2 - \frac{1}{2(1 + r_n)^2} K u_2 u_3, \\
\frac{du_3}{d\ell} &= \epsilon u_3 - \frac{3}{2(1 + r_n)^2} K u_3^2 - \frac{(N - 1)}{6(1 + r_s)^2} K u_2^2,
\end{align*}
\]  
where \( K \) is the phase space factor discussed in class.

(b) Show that these equations reduce to the expected equations in the limits corresponding to the models with \( O(N) \), Ising, and \( O(N - 1) \) symmetry just noted.

(c) Consider the fixed point of the flow equations with \( O(N) \) symmetry: \( r_s = r_n = r^* \), and \( u_1 = u_2 = u_3 = u^* \). Show that, to leading order in \( \epsilon \), this fixed point has two relevant eigenvalues \( 2 - (N + 2)\epsilon/(N + 8) \) and \( 2 - 2\epsilon/(N + 8) \) (see Plischke and Bergersen for some calculational hints).