1. Consider a single “quantum dot” of bosons described by the Hamiltonian

\[ H = -\mu \hat{n} + \frac{U}{2} \hat{n}(\hat{n} - 1) \]  

where \( \hat{n} = \hat{b}_a \hat{b}^\dagger_a \) (this is actually a reasonable description of certain superconducting quantum dots). The chemical potential, \( \mu \), is chosen so that the ground state has \( \langle \hat{n} \rangle = n_0 \), a non-zero integer.

(a) What is the allowed range of \( \mu \)?

(b) Compute the boson Green’s function

\[ G(\omega_n) = \int_0^{1/T} d\tau \langle \hat{b}(\tau)\hat{b}^\dagger(0) \rangle e^{i\omega_n \tau} \]

where \( \tau \) is imaginary time, and \( \omega_n \) is an integer multiple of \( 2\pi T \). Do this by inserting complete sets of exact eigenstates. Write down the spectral density associated with \( G \) at \( T = 0 \).

(c) Expand \( G(\omega_n) \) in powers of \( \omega_n \) to order \( \omega_n^2 \) at \( T = 0 \). For what value of \( \mu \) does the term linear in \( \omega \) vanish? Are there any special properties of the spectrum at this value of \( \mu \)?

2. We will determine the spectrum of normal modes of the action (10.21) in the superfluid phase with \( \tilde{r} < 0 \). The action has a saddle point at \( \Psi_B = \Psi_0 \neq 0 \) in this phase.

(a) What is the value of \( \Psi_0 \)? Choose \( \Psi_0 \) to be real and positive.

(b) Write \( \Psi_B(x, \tau) = (\Psi_0 + \psi_1(x, \tau)) e^{i\theta(x, \tau)} \) (\( \psi_1 \) and \( \theta \) are real), and expand the action to quadratic order in \( \psi_1 \) and \( \theta \). The terms linear in \( \psi_1 \) or linear in \( \theta \) will vanish after you use the condition on the periodicity of the fields. Retain the terms of order \( \psi_1^2, \theta^2 \) and \( \psi_1 \theta \) and express them in frequency and momentum space.

(c) The physical density of bosons is given by \( \partial \ln Z_B / \partial \mu \). Assume that \( \tilde{r} \) depends upon \( \mu \), and neglect the dependence of all other couplings on \( \mu \). Thus relate, to linear order, \( \psi_1 \) and \( \theta \) to the density fluctuations.

(d) Diagonalize the quadratic form in 2(b) to obtain the spectrum of excitations.