Quantum Phase Transitions

Subir Sachdev; email: subir.sachdev@yale.edu

Chapter 10: *Exercises*

1. Consider a single "quantum dot" of bosons described by the Hamiltonian

$$H = -\mu \hat{n} + \frac{U}{2} \hat{n}(\hat{n} - 1) \tag{1}$$

where $\hat{n} = \hat{b}^\dagger \hat{b}$ (this is actually a reasonable description of certain superconducting quantum dots). The chemical potential, μ, is chosen so that the ground state has $\langle \hat{n} \rangle = n_0$, a non-zero integer.

   (a) What is the allowed range of μ ?

   (b) Compute the boson Green's function

$$G(\omega_n) = \int_0^{1/T} d\tau \langle \hat{b}(\tau) \hat{b}^\dagger(0) \rangle e^{i\omega_n \tau} \tag{2}$$

   where τ is imaginary time, and ω_n is an integer multiple of $2\pi T$. Do this by inserting complete sets of exact eigenstates. Write down the spectral density associated with G at $T = 0$.

   (c) Expand $G(\omega_n)$ in powers of ω_n to order ω_n^2 at $T = 0$. For what value of μ does the term linear in ω vanish ? Are there any special properties of the spectrum at this value of μ ?

2. We will determine the spectrum of normal modes of the action (10.21) in the superfluid phase with $\tilde{r} < 0$. The action has a saddle point at $\Psi_B = \Psi_0 \neq 0$ in this phase.

   (a) What is the value of Ψ_0 ? Choose Ψ_0 to be real and positive.

   (b) Write $\Psi_B(x, \tau) = (\Psi_0 + \psi_1(x, \tau)) e^{i\theta(x,\tau)}$ (ψ_1 and θ are real), and expand the action to quadratic order in ψ_1 and θ. The terms linear in ψ_1 or linear in θ will vanish after you use the condition on the periodicity of the fields. Retain the terms of order ψ_1^2, θ^2 and $\psi_1 \theta$ and express them in frequency and momentum space.

   (c) The physical density of bosons is given by $\partial \ln Z_B / \partial \mu$. Assume that $\tilde{r}$ depends upon μ, and neglect the dependence of all other couplings on μ. Thus relate, to linear order, ψ_1 and θ to the density fluctuations.

   (d) Diagonalize the quadratic form in 2(b) to obtain the spectrum of excitations.