Quantum Phase Transitions
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Chapter 14: Exercises


1. In this problem we return to the Ising chain in a transverse field

\[ H_I = -\sum_n \left( \sigma_{2n}^z \sigma_{2n+2}^z + g \sigma_{2n}^x \right), \quad (1) \]

where, for further convenience, we have placed the Ising spins only on the even sites of a one-dimensional chain. We will establish the key result \( \dim[\sigma^z] = 1/8 \) at the critical point \( g = 1 \).

(a) **Duality:** First, we introduce a ‘dual’ formulation of \( H_I \). The operators in \( H_I \) are defined by the identities

\[
\begin{align*}
\sigma_n^z \sigma_n^z &= 1 \\
\sigma_n^x \sigma_n^x &= 1 \\
\sigma_n^z \sigma_n^x &= -\sigma_n^x \sigma_n^z \\
\sigma_m^x \sigma_n^z &= \sigma_n^x \sigma_m^z, \quad n \neq m
\end{align*}
\]

(2)

We now introduce dual operators, \( \overline{\sigma}_{2n+1}^z, \overline{\sigma}_{2n+1}^x \), residing on the odd sites, which are defined by

\[
\begin{align*}
\overline{\sigma}_{2n+1}^z &\equiv \sigma_{2n}^z \sigma_{2n+2}^z \\
\overline{\sigma}_{2n+1}^x &\equiv \prod_{m \leq n} \sigma_{2n}^x
\end{align*}
\]

(3)

Prove that the operators \( \overline{\sigma}_{2n+1}^z \) and \( \overline{\sigma}_{2n+1}^x \) obey exactly the same identities as the corresponding operators in (2); hence these operators can also be viewed as the Pauli matrices associated with a dual set of (fictitious) Ising spins. Also show that \( H_I \) can be rewritten in terms of these new spins as

\[ H_I = -\sum_n \left( g \overline{\sigma}_{2n-1}^z \overline{\sigma}_{2n+1}^z + \overline{\sigma}_{2n+1}^x \right), \quad (4) \]

Notice that (1) and (4) have precisely the same form with \( g \leftrightarrow 1/g \). Hence \( H_I \) is ‘self-dual’. Argue that this means that the critical point of \( H_I \) can only be at \( g = 1 \).
(b) **Doubling** Now we introduce another copy of $H_I$, denoted $H'_I$, defined in terms of a separate set of spins $\tau^x_{2n+1}$, $\tau^z_{2n+1}$ which reside on the odd sites, and which commute with all the $\sigma^x_{2n}$ (and hence also commute with the $\tau^x_{2n+1}$). So

$$H'_I = -\sum_n \left( \tau^z_{2n-1} \tau^z_{2n+1} + g \sigma^x_{2n} \right), \quad (5)$$

We can also introduce a dual representation of $H'_I$ in terms of Ising spins $\tau^z_{2n}$ as in (3).

(c) **Equivalence to XY model.** Finally, we introduce yet another set of operators, $\mu^z_{n+1/2}$, $\mu^z_{n+1/2}$, which reside on the sites in between those of the spins of $H_I$ and $H'_I$. These are defined by

$$
\begin{align*}
\mu^z_{2n+1/2} &\equiv \sigma^z_{2n} \tau^z_{2n+1} \\
\mu^z_{2n-1/2} &\equiv \tau^z_{2n-1} \sigma^z_{2n} \\
\mu^z_{2n+1/2} &\equiv \tau^z_{2n} \sigma^z_{2n+1} \\
\mu^z_{2n-1/2} &\equiv \sigma^z_{2n-1} \tau^z_{2n}
\end{align*}
\quad (6)$$

Prove that $\mu^z_{n+1/2}$ also obey the identities in (2), and so can also be considered as a set of Pauli matrices for $S = 1/2$ spins residing on the half-integer sites. Show also that

$$H_{XY} \equiv H_I + H'_I = \sum_n \left( \mu^z_{n-1/2} \mu^z_{n+1/2} + g \mu^z_{n-1/2} \mu^z_{n+1/2} \right) \quad (7)$$

(d) Notice that at $g = 1$, $H_{XY}$ is precisely the quantum ‘XX’ model analyzed in Section 14.1. We established there that $\dim[\mu^z] = 1/4$ at $g = 1$. Deduce from this that $\dim[\sigma^z] = 1/8$.

2. **Interacting bosons in one dimension**: We consider the following Hamiltonian describing interacting bosons in the continuum in the presence of an applied periodic potential:

$$H = \int dx \left[ \Psi^\dagger(x) \left( -\frac{1}{2m} \frac{\partial^2}{\partial x^2} - \mu \right) \Psi(x) + V_G \cos(Gx) \rho(x) \right] + \frac{1}{2} \int dx dx' \rho(x) V_I(x-x') \rho(x') \quad (8)$$

where the number density $\rho(x) = \Psi^\dagger(x) \Psi(x)$, the operators obey the canonical commutation relations

$$[\Psi(x), \Psi^\dagger(x)] = \delta(x-x'), \quad (9)$$

2
$V_l(x)$ is some short-range repulsive interaction, and $V_G$ is an externally applied periodic potential. The chemical potential, $\mu$, is chosen so that the mean number density of the bosons $\langle \rho(x) \rangle = \rho_0$.

(a) We expect that the ground state of $H$ is superfluid-like (more precisely, we will show below that the superfluid order is only quasi-long-range) for generic values of $\rho_0$; there are exceptions when $\rho_0$ satisfies certain commensuration conditions, and these will be discussed below. For such ‘superfluid’ states, argue that the low energy properties of $H$ are actually described by the Tomonaga-Luttinger liquid model in (14.22-14.26); the parameters $v_F$ and $K$ depend upon the $m$, $\rho_0$, $V_G$, and $V_l(x)$ in a highly non-trivial manner, and this dependence is usually impossible to determine exactly. (I mention, as an aside, that for Galilean-invariant systems we have the exact result $Kv_F = \frac{\pi \rho_0}{m}$). The argument proceeds as follows. Imagine discretizing $H$ on a lattice of spacing $a$, where $a$ is much smaller than $1/G$ and the mean spacing between the bosons. Because of the repulsive interaction between the bosons, it is highly unlikely that more than one boson will ever occupy any given lattice site. Let us label the sites by the integer, $i$, and introduce the lattice boson operator, $b_i$, obeying $[b_i, b_j^\dagger] = \delta_{ij}$. So $b_i^\dagger b_i = 0, 1$, which is the “hard-core” condition. As in the discussion of the Jordan-Wigner transformation, show that we can make a correspondence between the states of the $S = 1/2$ spin and the hard-core bosons. Show that the operator correspondence is now very simple, and there is no need for any non-local “string” operators:

$$\sigma^z_i = b_i^\dagger b_i - 1.$$  \hspace{1cm} (10)

Show that the interacting boson model (for $V_G = 0$) then maps onto a $S = 1/2$ spin chain similar that in (14.1). What is the interpretation of $\mu$ in the spin language?

(b) Continuing the discussion of the $V_G = 0$ case, use the arguments in Chapter 14 to deduce the following correspondences between the operators of $H$ and those of the TL liquid:

$$\Psi^\dagger(x) = e^{-i\theta(x)} \sum_{n=-\infty}^{\infty} A_n e^{i2\pi \rho_0 nx + i2n\phi(x)}$$

$$\rho(x) = \frac{1}{\pi} \partial_x \phi(x) + \sum_{n\neq 0} B_n e^{i2\pi \rho_0 nx + i2n\phi(x)}, \hspace{1cm} (11)$$
for some constants $A_n$, $B_n$. Use this to compute the long-distance form of
$(\Psi^\dagger(x)\Psi(x'))$, and thus establish that the superfluid-order is only quasi-long-
range.

(c) Now consider the consequences of $V_G \neq 0$. Use the representation of the density
in (11) to examine the structure of the perturbation theory for the partition
function in powers of $V_G$. Show that the spatial integral causes most terms in this
perturbation theory to average out to zero on a short distance scale (of order $1/G$
or the mean particle spacing) unless

$$\frac{2\pi \rho_0}{G} = \frac{p}{q},$$

where $p$ and $q$ are relatively prime integers. For these special values of $\rho_0$, show
that the leading perturbation to $S_{TL}$ is described by the action

$$S = S_{TL} - v \int dx d\tau \cos(2q\phi(x, \tau))$$

where $v \sim V_G^p$. For what range of values of $K$ is the superfluid state stable \textit{i.e.}
v is an irrelevant perturbation? For $K$ outside this range, we expect that $v$ will
renormalize to large values – this will pin the allowed values of $\phi$, and the resulting
ground state will be a Mott insulator. Relate the commensuration condition on
the allowed values of $\rho_0$ at which such Mott insulators can arise to the earlier
discussion in the chapter on the boson Hubbard model. What is the exponent
characterizing the power-law decay of $(\Psi^\dagger(x)\Psi(x'))$ in the superfluid state just
before it becomes unstable to the Mott insulator?