Quantum Phase Transitions

Subir Sachdev; email: subir.sachdev@yale.edu

Chapter 14: *Exercises*

- Useful reference for a complete treatment of bosonization: J. von Delft and H. Schoeller, Annalen der Physik, **4**, 225 (1999); cond-mat/9805275.

1. In this problem we return to the Ising chain in a transverse field

$$H_I = -\sum_n \left(\sigma^z_{2n}\sigma^z_{2n+2} + g\sigma^x_{2n} \right), \tag{1}$$

where, for furture convenience, we have placed the Ising spins only on the *even* sites of a one-dimensional chain. We will establish the key result $\dim[\sigma^z] = 1/8$ at the critical point $g = 1$.

(a) **Duality**: First, we introduce a 'dual' formulation of H_I. The operators in H_I are defined by the identities

$$\begin{aligned} \sigma^z_n\sigma^z_n &= 1 \\ \sigma^x_n\sigma^x_n &= 1 \\ \sigma^z_n\sigma^x_n &= -\sigma^x_n\sigma^z_n \\ \sigma^z_n\sigma^x_m &= \sigma^x_m\sigma^z_n \quad , \quad n \neq m \end{aligned} \tag{2}$$

We now introduce dual operators, $\overline{\sigma}^z_{2n+1}$, $\overline{\sigma}^x_{2n+1}$, residing on the *odd* sites, which are defined by

$$\begin{aligned} \overline{\sigma}^x_{2n+1} &\equiv \sigma^z_{2n}\sigma^z_{2n+2} \\ \overline{\sigma}^z_{2n+1} &\equiv \prod_{m\leq n} \sigma^x_{2n} \end{aligned} \tag{3}$$

Prove that the operators $\overline{\sigma}^x_{2n+1}$ and $\overline{\sigma}^z_{2n+1}$ obey exactly the same identities as the corresponding operators in (2); hence these operators can also be viewed as the Pauli matrices associated with a dual set of (fictitious) Ising spins. Also show that H_I can be rewritten in terms of these new spins as

$$H_I = -\sum_n \left(g\overline{\sigma}^z_{2n-1}\overline{\sigma}^z_{2n+1} + \overline{\sigma}^x_{2n+1} \right), \tag{4}$$

Notice that (1) and (4) have precisely the same form with $g \leftrightarrow 1/g$. Hence H_I is 'self-dual'. Argue that this means that the critical point of H_I can only be at $g = 1$.

(b) **Doubling** Now we introduce another copy of H_I, denoted H_I', defined in terms of a separate set of spins τ_{2n+1}^x, τ_{2n+1}^z which reside on the odd sites, and which commute with all the $\sigma_{2n}^{x,z}$ (and hence also commute with the $\overline{\sigma}_{2n+1}^{x,z}$). So

$$H_I' = -\sum_n \left(\tau_{2n-1}^z \tau_{2n+1}^z + g\sigma_{2n}^x \right), \tag{5}$$

We can also introduce a dual representation of H_I' in terms of Ising spins $\overline{\tau}_{2n}^{z,x}$ as in (3).

(c) **Equivalence to XY model**. Finally, we introduce yet another set of operators, $\mu_{n+1/2}^x$, $\mu_{n+1/2}^z$, which reside on the sites in between those of the spins of H_I and H_I'. These are defined by

$$\begin{aligned}
\mu_{2n+1/2}^z &\equiv \sigma_{2n}^z \tau_{2n+1}^z \\
\mu_{2n-1/2}^z &\equiv \tau_{2n-1}^z \sigma_{2n}^z \\
\mu_{2n+1/2}^x &\equiv \overline{\tau}_{2n}^z \overline{\sigma}_{2n+1}^z \\
\mu_{2n-1/2}^x &\equiv \overline{\sigma}_{2n-1}^z \overline{\tau}_{2n}^z
\end{aligned} \tag{6}$$

Prove that $\mu_{n+1/2}^{x,z}$ also obey the identities in (2), and so can also be considered as a set of Pauli matrices for $S = 1/2$ spins residing on the half-integer sites. Show also that

$$H_{XY} \equiv H_I + H_I' = \sum_n \left(\mu_{n-1/2}^z \mu_{n+1/2}^z + g\mu_{n-1/2}^x \mu_{n+1/2}^x \right) \tag{7}$$

(d) Notice that at $g = 1$, H_{XY} is precisely the quantum 'XX' model analyzed in Section 14.1. We established there that $\dim[\mu^z] = 1/4$ at $g = 1$. Deduce from this that $\dim[\sigma^z] = 1/8$.

2. **Interacting bosons in one dimension**: We consider the following Hamiltonian describing interacting bosons in the continuum in the presence of an applied periodic potential:

$$H = \int dx \left[\Psi^\dagger(x) \left(-\frac{1}{2m}\frac{\partial^2}{\partial x^2} - \mu \right) \Psi(x) + V_G \cos(Gx)\rho(x) \right] + \frac{1}{2} \int dx dx' \rho(x) V_I(x-x') \rho(x') \tag{8}$$

where the number density $\rho(x) = \Psi^\dagger(x)\Psi(x)$, the operators obey the canonical commutation relations

$$[\Psi(x), \Psi^\dagger(x)] = \delta(x - x'), \tag{9}$$

$V_I(x)$ is some short-range repulsive interaction, and V_G is an externally applied periodic potential. The chemical potential, μ, is chosen so that the mean number density of the bosons $\langle \rho(x) \rangle = \rho_0$.

(a) We expect that the ground state of H is superfluid-like (more precisely, we will show below that the superfluid order is only quasi-long-range) for generic values of ρ_0; there are exceptions when ρ_0 satisfies certain commensuration conditions, and these will be discussed below. For such 'superfluid' states, argue that the low energy properties of H are actually described by the Tomonaga-Luttinger liquid model in (14.22-14.26); the parameters v_F and K depend upon the m, ρ_0, V_G, and $V_I(x)$ in a highly non-trivial manner, and this dependence is usually impossible to determine exactly. (I mention, as an aside, that for Galilean-invariant systems we have the exact result $K v_F = \pi \rho_0 / m$). The argument proceeds as follows. Imagine discretizing H on a lattice of spacing a, where a is much smaller than $1/G$ and the mean spacing between the bosons. Because of the repulsive interaction between the bosons, it is highly unlikely that more than one boson will ever occupy any given lattice site. Let us label the sites by the integer, i, and introduce the lattice boson operator, b_i, obeying $[b_i, b_j^\dagger] = \delta_{ij}$. So $b_i^\dagger b_i = 0, 1$, which is the "hard-core" condition. As in the discussion of the Jordan-Wigner transformation, show that we can make a correspondence between the states of the $S = 1/2$ spin and the hard-core bosons. Show that the operator correspondence is now very simple, and there is no need for any non-local "string" operators:

$$\sigma_i^+ = b_i^\dagger \quad ; \quad \sigma_i^z = 2 b_i^\dagger b_i - 1. \tag{10}$$

Show that the interacting boson model (for $V_G = 0$) then maps onto a $S = 1/2$ spin chain similar that in (14.1). What is the interpretation of μ in the spin language ?

(b) Continuing the discussion of the $V_G = 0$ case, use the arguments in Chapter 14 to deduce the following correspondences between the operators of H and those of the TL liquid:

$$\begin{aligned}
\Psi^\dagger(x) &= e^{-i\theta(x)} \sum_{n=-\infty}^{\infty} A_n e^{i2\pi\rho_0 n x + i2n\phi(x)} \\
\rho(x) &= \frac{1}{\pi}\partial_x \phi(x) + \sum_{n \neq 0} B_n e^{i2\pi\rho_0 n x + i2n\phi(x)},
\end{aligned} \tag{11}$$

for some constants A_n, B_n. Use this to compute the long-distance form of $\langle \Psi^\dagger(x)\Psi(x')\rangle$, and thus establish that the superfluid-order is only quasi-long-range.

(c) Now consider the consequences of $V_G \neq 0$. Use the representation of the density in (11) to examine the structure of the perturbation theory for the partition function in powers of V_G. Show that the spatial integral causes most terms in this perturbation theory to average out to zero on a short distance scale (of order $1/G$ or the mean particle spacing) unless

$$\frac{2\pi\rho_0}{G} = \frac{p}{q}, \qquad (12)$$

where p and q are relatively prime integers. For these special values of ρ_0, show that the leading perturbation to $\mathcal{S}_{TL}$ is described by the action

$$\mathcal{S} = \mathcal{S}_{TL} - v \int dx d\tau \cos(2q\phi(x,\tau)) \qquad (13)$$

where $v \sim V_G^p$. For what range of values of K is the superfluid state stable *i.e.* v is an irrelevant pertubation ? For K outside this range, we expect that v will renormalize to large values – this will pin the allowed values of ϕ, and the resulting ground state will be a Mott insulator. Relate the commensuration condition on the allowed values of ρ_0 at which such Mott insulators can arise to the earlier discussion in the chapter on the boson Hubbard model. What is the exponent characterizing the power-law decay of $\langle \Psi^\dagger(x)\Psi(x')\rangle$ in the superfluid state just before it becomes unstable to the Mott insulator ?